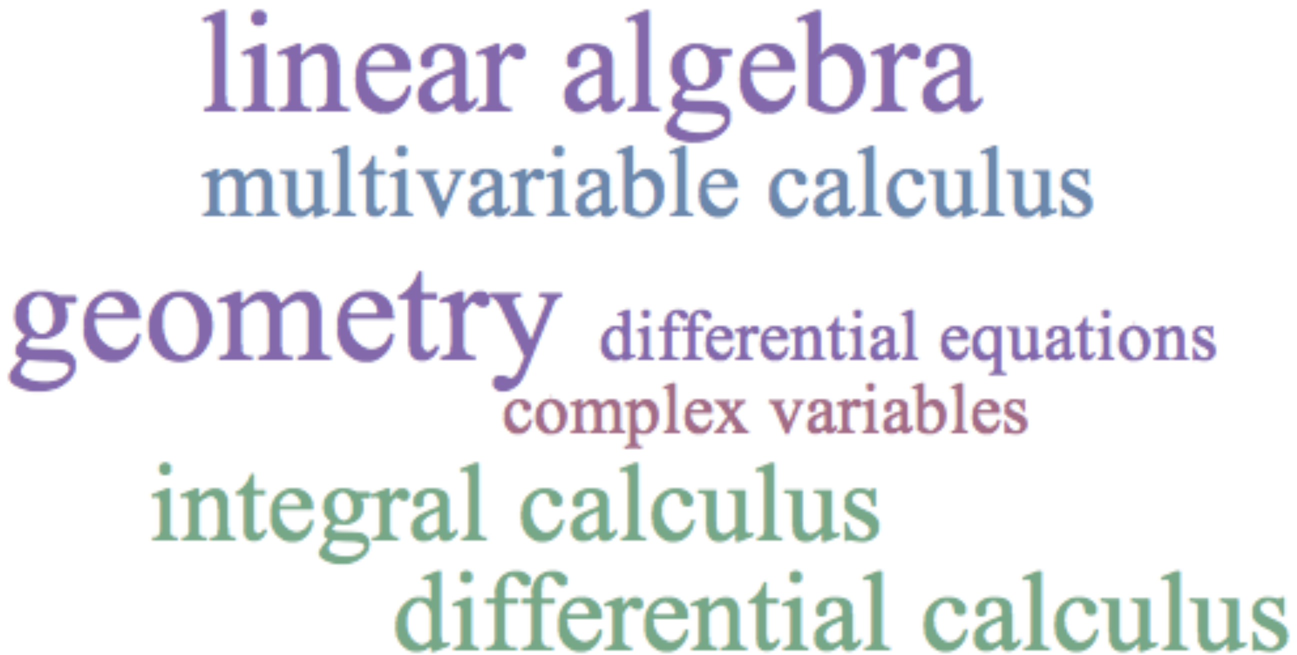
# Engaging Applications in Differential Geometry using Maple

Kristen R. Schreck
Saint Xavier University
schreck@sxu.edu

ICTCM 2015

# Differential Geometry



# *A First Course in Differential Geometry*

- Perfect transition course from calculus, linear algebra and differential equations to higher mathematics
- Intuitive, interdisciplinary, and interesting!
- Visualizations using Maple go hand-in-hand with the understanding of the mathematics
- Focus on comprehending the mathematics behind the constructions in Maple rather than simply drawing pictures
- Emphasize the compelling history of the subject to make the study complete

# *Maple Applications & Projects*

- Geometry of Plane and Space Curves
  - Involutes
  - Evolutes
  - Osculating Circle and Sphere

- Curvature
  - Normal & Principal
  - Gaussian & Mean

- Surfaces
  - Ruled
  - Developable (Gaussian curvature = 0)

# *Maple Applications & Projects*

- Minimal Surfaces
  - Constant Mean Curvature
  - The Gauss Map

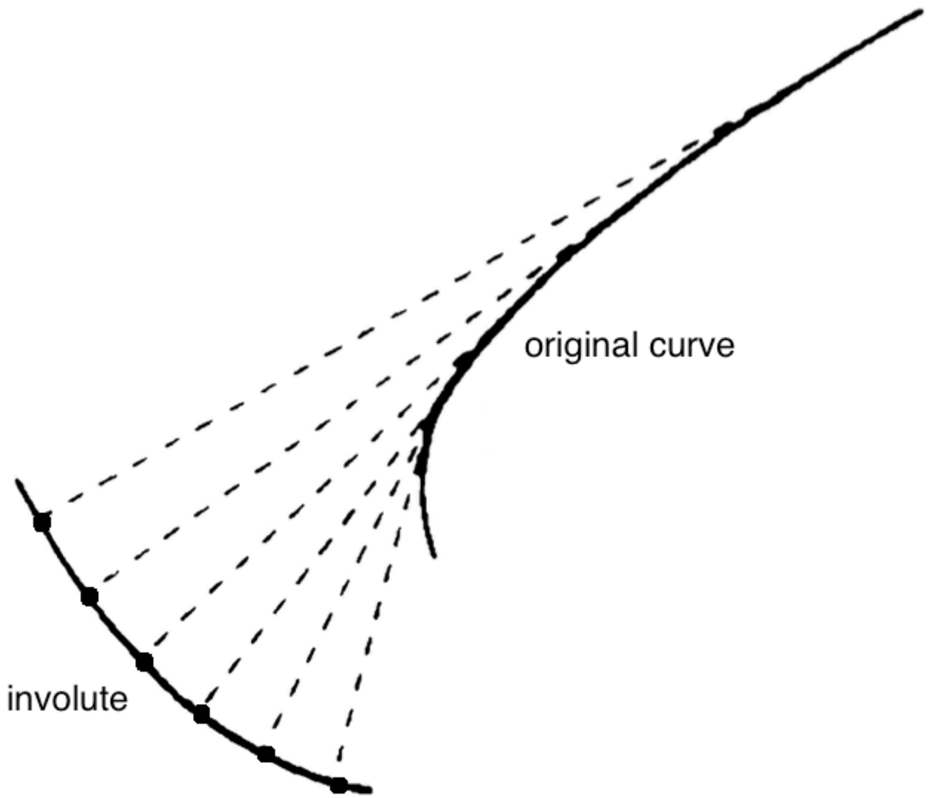
- Geodesics on Surfaces

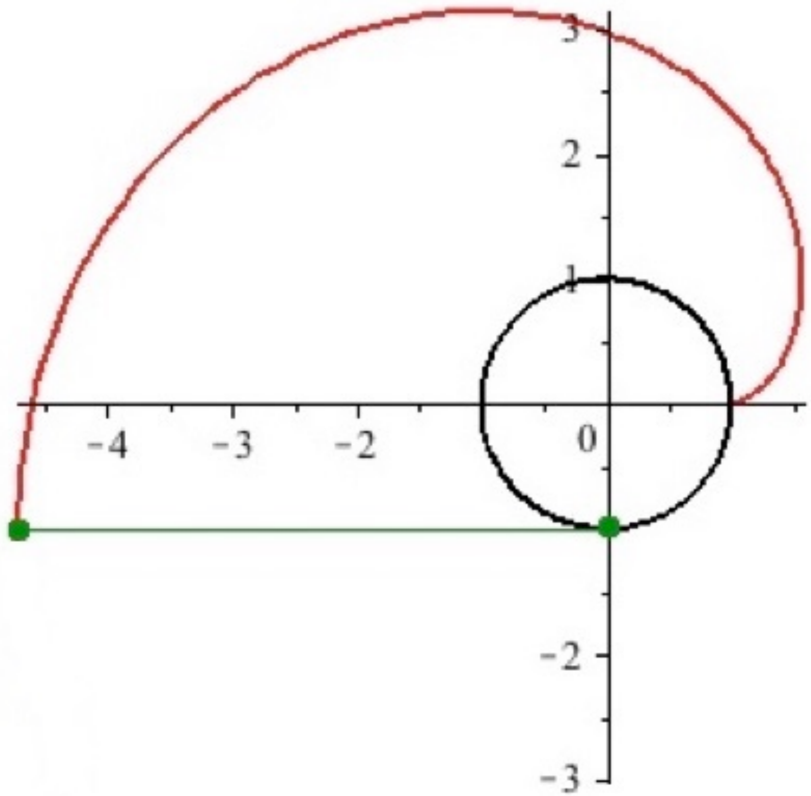
- Gauss-Bonnet Theorem

# Huygens & Plane Curves (17$^{th}$ C.)

- Christiaan Huygens (1629-1695), inventor of the pendulum clock
- Curvature was a key concept
- Huygens introduced the **involute** of a planar curve
- Important technical achievement, big improvement in time keeping

# Involutes of Plane and Space Curves



# Maple

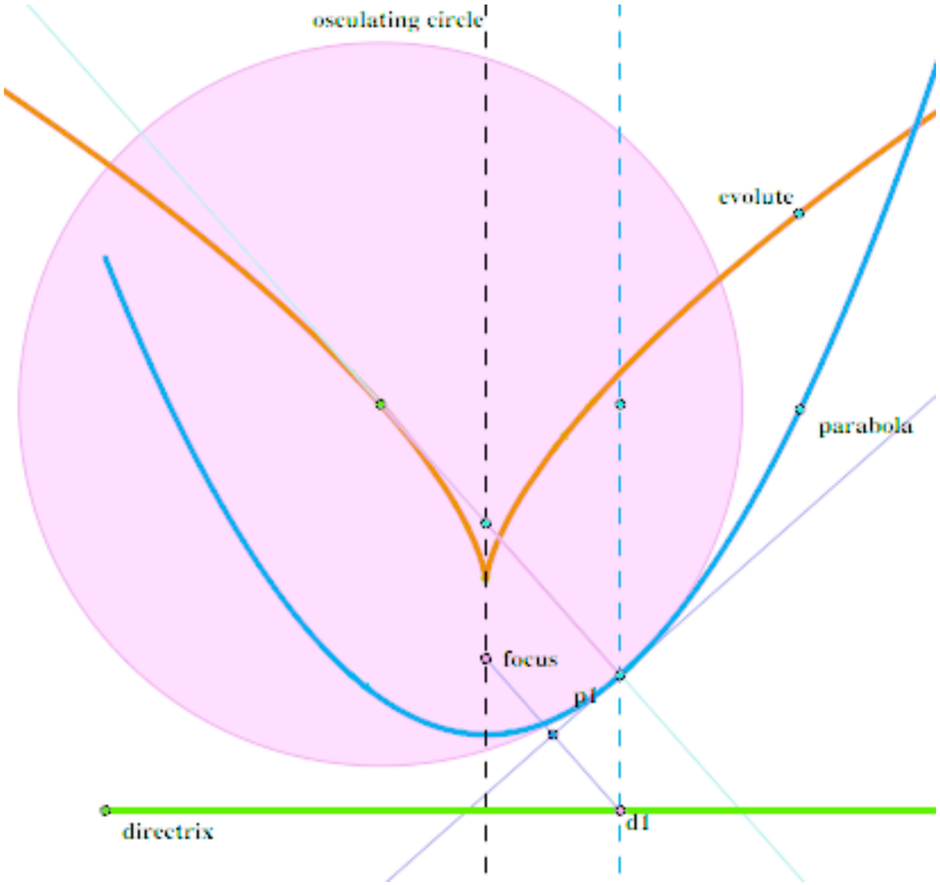
Involute Program

# Clairaut & Space Curves (18th C.)

- Alexis Claude Clairaut (1713-1765)
- First comprehensive analysis of space curves, *Recherches sur les courbes à double courbure* (1731)
- Introduced the phrase "courbe a double courbure", became commonly recognized terminology for *space curve*

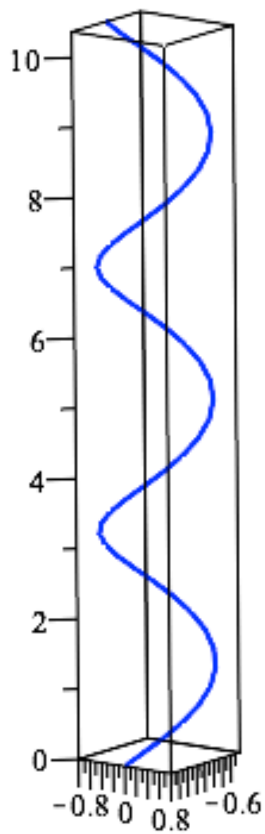
# Evolute of a Plane Curve

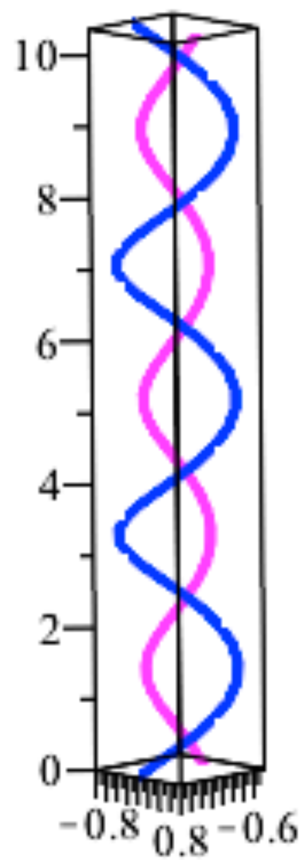
- Dual notion to the involute is called an **evolute**
- The normal lines to the **involute** at two successive points will meet in a point called the **center of curvature**
- The locus of the centers of curvature is called the **evolute**



# Evolute of a Space Curve – A Helix

$$\alpha(t) \qquad E(t) = \alpha(t) + \frac{1}{\kappa}N(t) + \frac{1}{\tau\kappa'}B(t)$$



# Monge & Descriptive Geometry ( 18th C.)

- Gaspard Monge (1746 – 1818)
- Reinvented the subject of descriptive geometry, used to train draftsmen, architects, engineers, and also mathematicians
- Developed a generalized theory and technical drawing method for representing three-dimensional figures in two dimensions
- Common problem: determine the projection of intersections of curved surfaces
- Ideas of evolutes of spaces curves, ruled surfaces, radius of curvature, osculating plane were outgrowths of Monge's work

# *Maple*

## Evolute Program

# Viviani's Curve

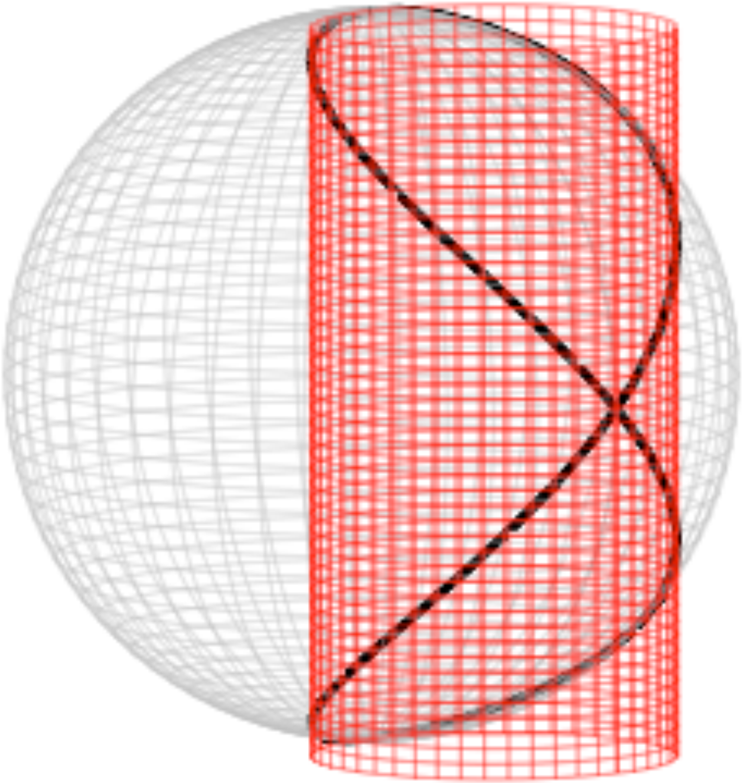
- Intersection of the cylinder

$$(x-a)^2 + y^2 = a^2$$

and the sphere

$$x^2 + y^2 + z^2 = 4a^2$$

- Studied in 1692 by Vincenzo Viviani, a distinguished student of Galileo



$$\alpha(t) = \left( a(1+\cos t), a\sin t, 2a\sin\left(\frac{t}{2}\right)\right)$$

# Maple

## Evolute of Viviani's Curve

# Monge & Differential Geometry (18th C.)

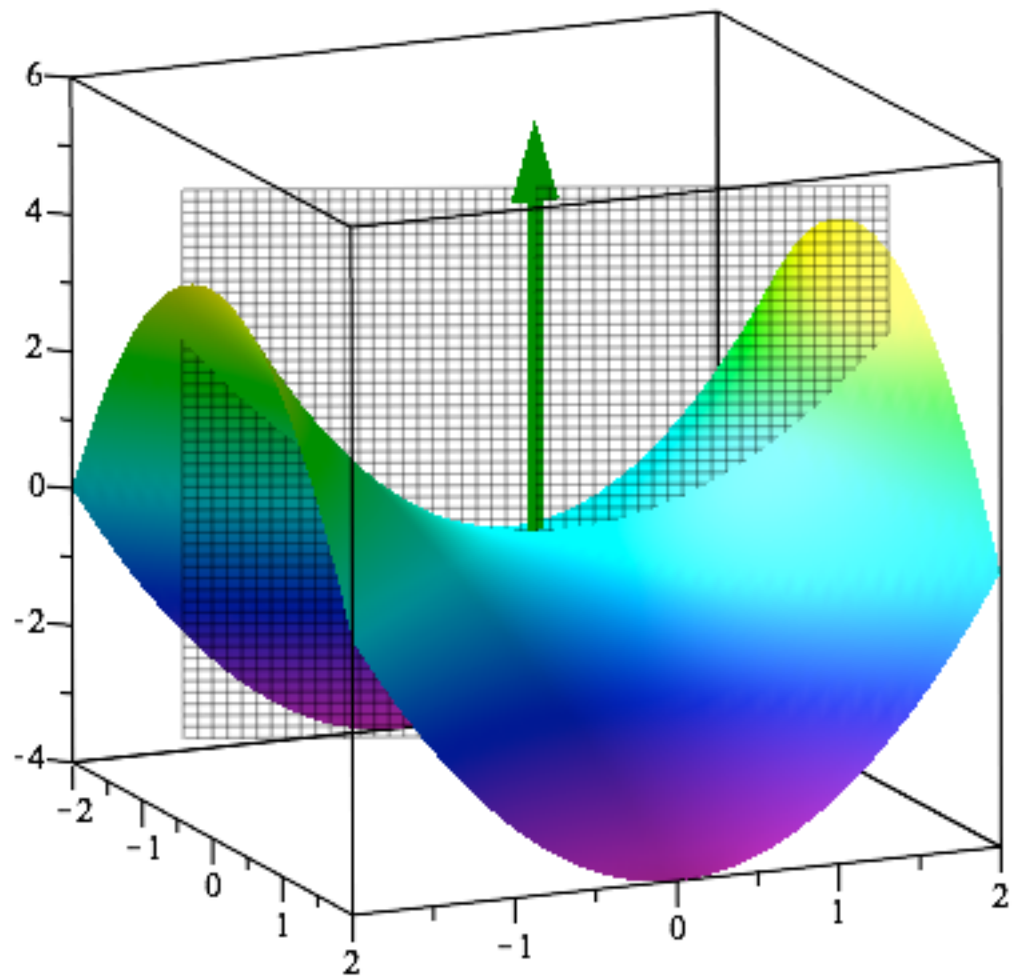
- Pioneered the field of differential geometry
- *Application de L'Analyse à la Géométrie* (1807), was the first textbook of its kind on differential geometry of curves and surfaces
- Monge combined geometric reasoning with calculus and analytic geometry of space which emphasized the theory of differential equations in solving problems
- Exemplary teacher at the École Polytechnique, taught his students to examine a single problem from three aspects: the analytical, the geometrical, and the practical

# *Euler & Curvature of Space Curves and Surfaces (18th C.)*

- Leonhard Euler (1707 – 1783)
- First to study curvature of surfaces
- Significant paper "Recherches sur la courbure des surfaces" (1760) showed method for finding the curvature of the surface at a point
- Determined the maximum and minimum values of curvature, $k_1$ and $k_2$ called the *principal curvatures*

# Animation of Normal Sections

- K(p) > 0
- K(p) < 0
- K(p) = 0



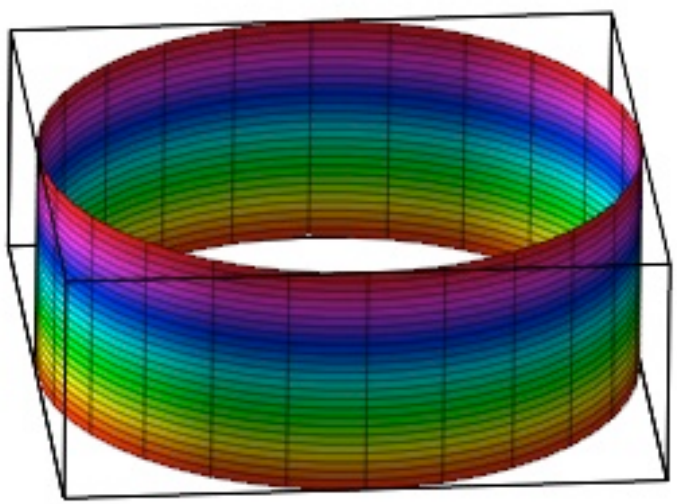
# Maple

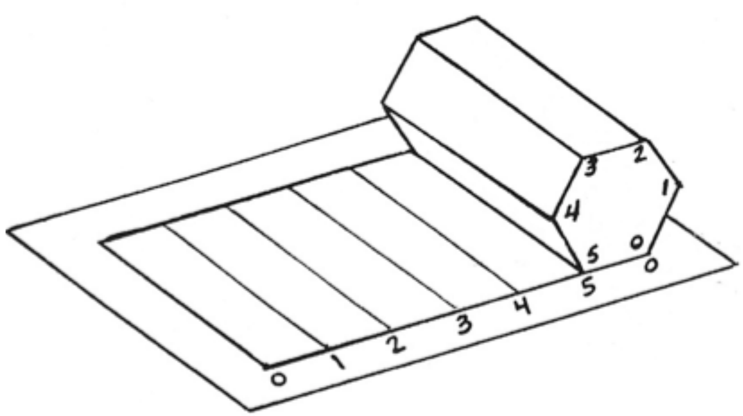
## Principal Curvatures

## &

## Animation of Normal Sections

# Ruled & Developable Surfaces

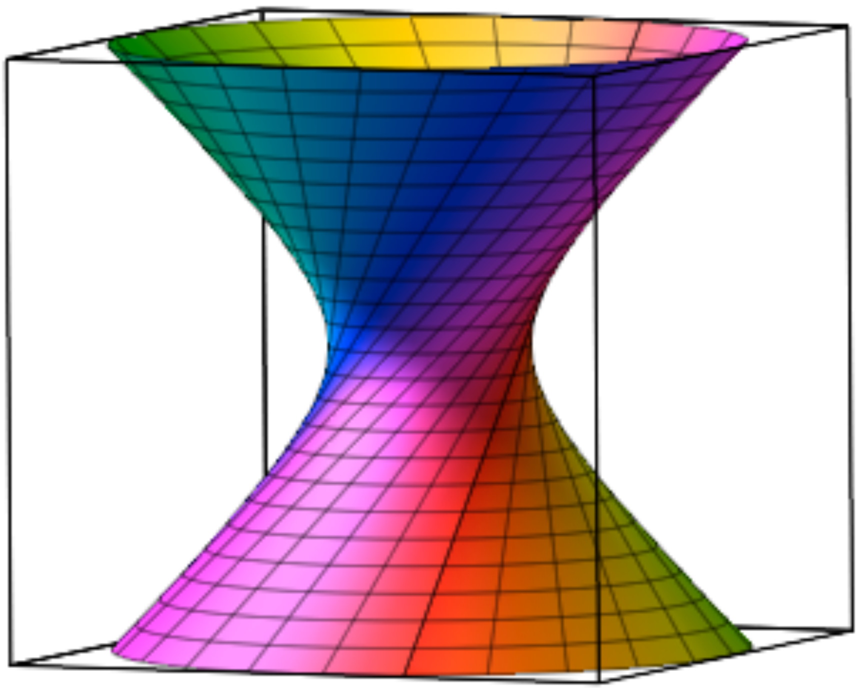


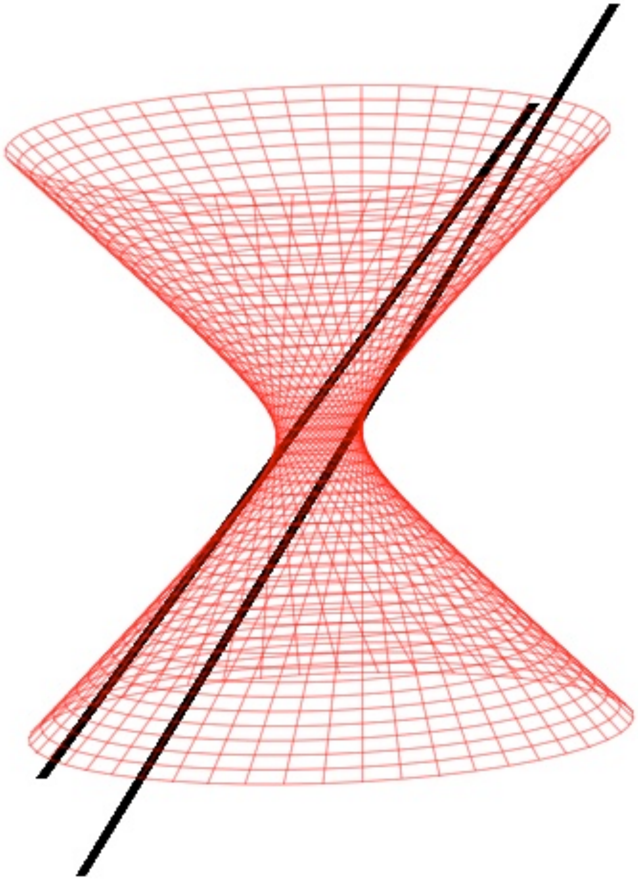



- A *ruled surface* is a surface swept out by a moving straight line in $R^3$
- *Ruled surfaces* are classified either as *developable* or *warped*
- A *developable surface* can be unrolled onto a plane and will lie flat, whereas, a *warped surface* will not

# Warped Surfaces

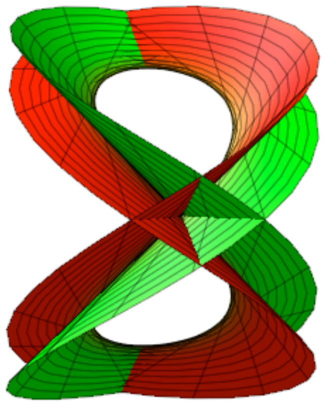
- Hyperboloid of revolution with skew lines as generators



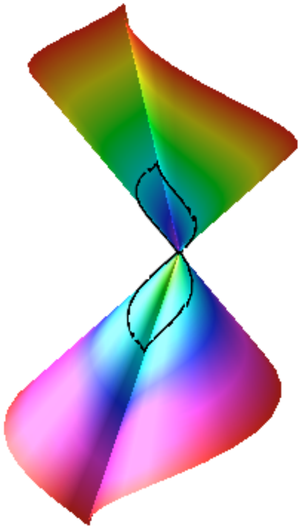


# Tangent and Polar Developable Surfaces

- Developable surfaces – first investigated in descriptive geometry since it was possible to construct them using purely geometrical means



- Tangent developable surface – a ruled surface generated by the tangent lines to a space curve, t tangent plane is constant along each generating line



- Polar developable surface – a surface enveloped by the normal planes of a space curve

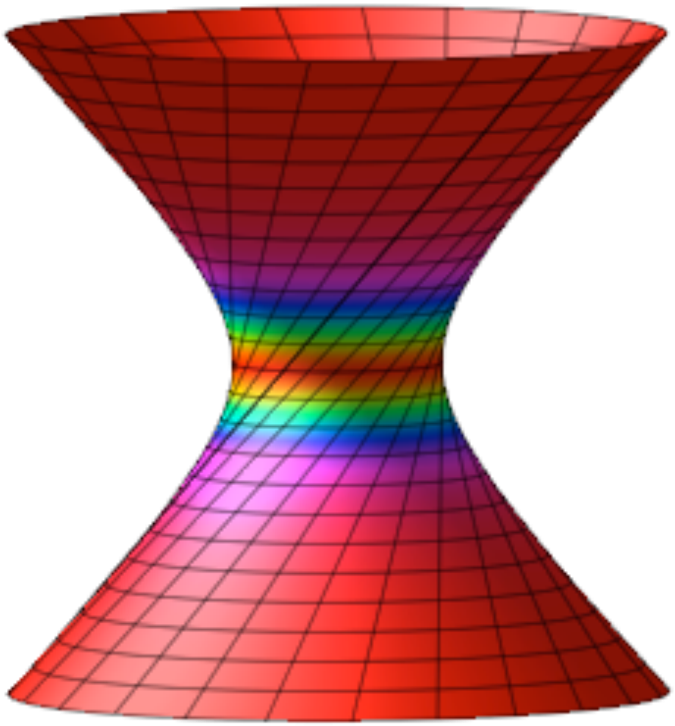
# Maple

## Ruled and Developable Surfaces

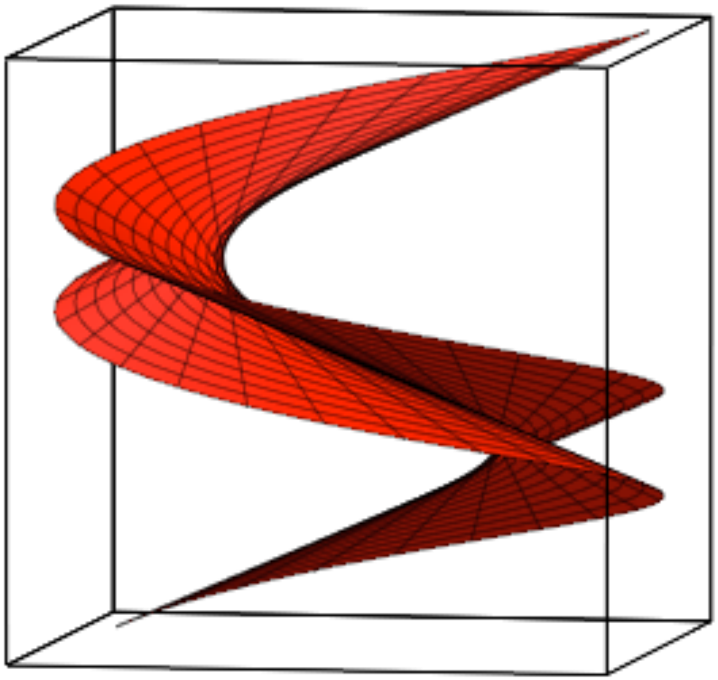
# *Gauss & Curvature (19th Century)*


- Carl Friedrich Gauss (1777-1855)
- Discovered one of the most renowned theorems of the 19th Century, his "great theorem", *Theorema Egregium*
- Proved that, although each principal curvatures $k_1$ and $k_2$ individually depend on the extrinsic properties of the surface (i.e., how the surface bends), the product of the two $K = k_1 k_2$ depends only on the intrinsic geometry of the surface

# Surfaces Colored by Gaussian Curvature



$$\kappa = 0$$



# *Maple*

## Surfaces Colored by Gaussian Curvature

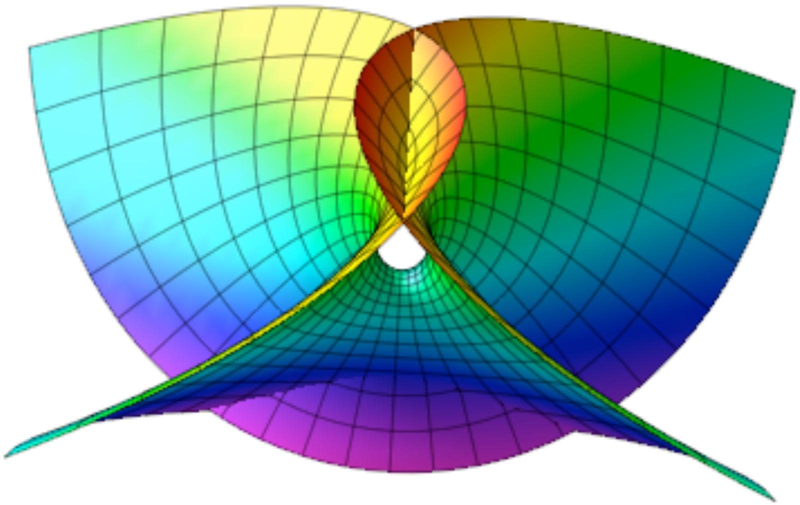
# Minimal Surfaces

- A surface that locally minimizes its area
- Equivalently, a surface whose *mean curvature* is zero

$$\frac{\kappa_1 + \kappa_2}{2} = 0$$

- Study originated in 1762 with Joseph-Louis LaGrange (1736-1813)
  and
  Jean Baptiste Meusnier (1754-1793) in 1776

- Enneper's Surface



$$Enneper := \left\langle u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + vu^2, u^2 - v^2 \right\rangle$$

$$MK(Enneper) = 0$$

# *Maple*

## Surfaces Colored by Mean Curvature

## &

## Comparing Gaussian and Mean Curvature

Thank you for attending !
&
Happy 03.14.15 Day!

(Putting the ***pair of pants surface*** to good use.)

# References


- Theodore Shifrin, *Differential Geometry: A First Course in Curves and Surfaces*, 2013
- John McCleary, *Geometry from a Differentiable Viewpoint*, 2012
- Thomas Banchoff and Stephen T. Lovett , *Differential Geometry of Curves and Surfaces*, 2010
- John Oprea, *Differential Geometry and Its Applications*, 2007
- Alfred Gray, *Modern Differential Geometry of Curves and Surface with Mathematica*, 2006
- Manfredo P. Do Carmo, *Differential Geometry*, 1976
- Dirk J. Struik, *Differential Geometry*, 1950
- Gaspard Monge, *Application de L'Analyse à la Géométrie*, 1805
- Kristen R. Schreck, *Monge's Legacy of Descriptive and Differential Geometry*, manuscript submitted for publication, Docent Press, 2015

$$\frac{\text{singerine}}{\text{cosgerine}} = $$

sound effect